Calculation of the Coëfficiënts of a Lattice Wave Digital Filter

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This report lists the formulaes to calculate the adaptor coëfficiënts for discrete time lattice wave digital filters realizing lowpass or bandpass transfer functions. The transfer functions generally consist of a rational function given by a numerator and a denominator polynomial, given is the time continuous complex variable s. Note that only odd order lowpass filters can be realized with lattice structures, while bandpass filters should be of even order (e.g. the order of the denominator should be odd or even, respectively).

The lattice filter itself is realized as the sum (lowpass version, see Figure 1) or the difference (bandpass version, Figure 2) of two allpass discete time filter sections. In the lowpass version, one of the sections is a cascade of a first order allpass section, all or not augmented with 2nd order all pass sections, while the other arm is built with 2nd order allpass sections only. The bandpass version consists of 2nd order sections in both arms.

It is assumed that stopband notches are integral parts of the transfer function, which means that their frequencies influence the approximation method used to obtain the description of the passband. Numerator and denominator polynomials thus are not independant from each other (this excludes e.g. a bandpass filter with cascaded separate notch filters).

For obtaining the coëfficiënts γ_i of a lattice wave digital filter structure, only the denominator polynomial will be needed. More in detail: the roots of the denominator polynomial in the time-continuous domain.

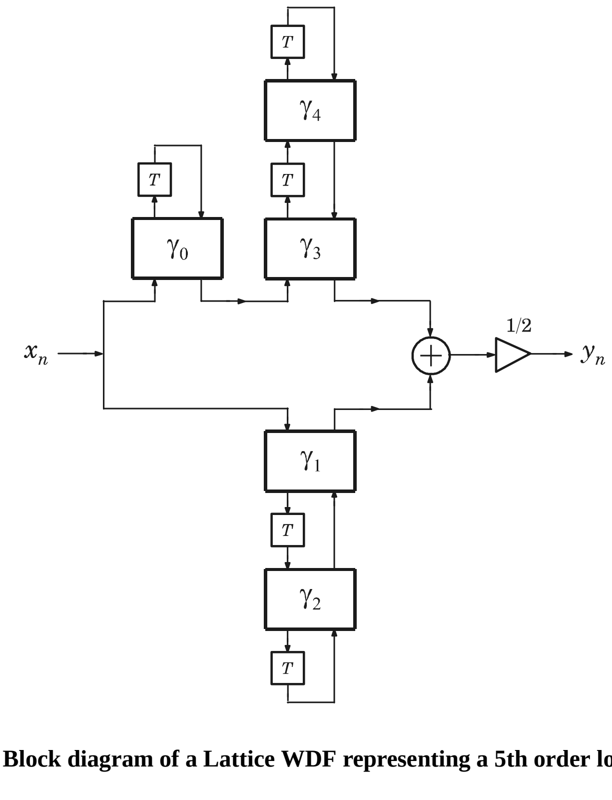

Figure 1: Block diagram of a Lattice WDF representing a 5th order lowpass filter.

Derivation of the γ's from the roots of $g(s)$

Suppose $H(s) = \dfrac{f(s)}{g(s)}$ describes a *lowpass* filter function. Then, since the filter order N is odd, we can write

$$H(s) = \frac{\text{Numerator Polynomial}}{(s-p_0)\displaystyle\prod_{i=1}^{(N-1)/2}(s-p_i)(s-\overline{p_i})} \qquad \text{with} \quad p_i = x_i + jy_i,\ \overline{p_i} = x_i - jy_i$$

For the single real root p_0, we need a first order all pass section with

$$\gamma_0 = \frac{1-b_0}{1+b_0}$$

where $b_0 = sign(b_0) \cdot p_0$

with $sign(b_0) = -1$, except for $N = 1 + n \cdot 4$ ($n = 1, 2, 3, ...$), for which N's $sign(b_0) = +1$

The complex conjugated roots can be realized with second order all pass sections. If we write

$$(s-p_i)(s-\overline{p_i}) = s^2 - a_i s + b_i$$

$$p_i = \text{real}(p_i) \pm j\ \text{imag}(p_i)$$

then

$$a_i = 2 \cdot |\text{real}(p_i)|$$

Note: since the roots are situated in the negative half plane, we need the absolute value here

$$b_i = \{\text{real}(p_i)\}^2 + \{\text{imag}(p_i)\}^2$$

This results in

$$\gamma_{2i-1} = \frac{a_i - b_i - 1}{a_i + b_i + 1}$$

$$\gamma_{2i} = \frac{1-b_i}{1+b_i}$$

Now γ_0 together with γ_3,γ_4 and γ_7,γ_8 etc. constitute the upper arm, while γ_1,γ_2 and γ_5,γ_6 etc make up the lower arm of the lattice structure.

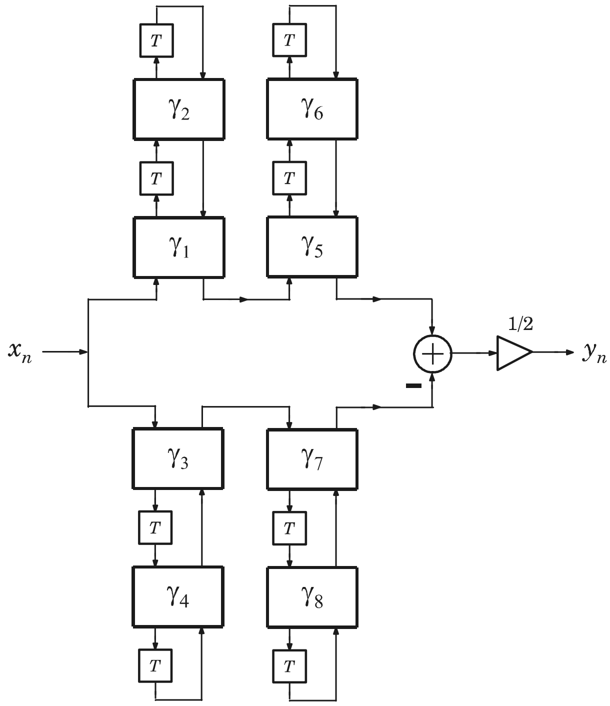

Figure 2: Block diagram of a Lattice WDF representing an 8th order bandpass filter.

For even order *bandpass* filters, $H(s)$ can be described with

$$H(s) = \frac{\text{Numerator Polynomial}}{\prod_{i=1}^{N/2}(s-p_i)(s-p_i^*)}$$

Here we only have to deal with the complex conjugated roots, thus

$$(s-p_i)(s-\overline{p_i}) = s^2 - a_i s + b_i$$

The results also will be

$$\gamma_{2i-1} = \frac{a_i - b_i - 1}{a_i + b_i + 1}$$

$$\gamma_{2i} = \frac{1 - b_i}{1 + b_i}$$

The upper arm consists of the cascade of γ_1, γ_2 and γ_5, γ_6 etc, with γ_3,γ_4 and γ_7,γ_8 in the lower arm.

Reconstruction of $H(z)$ when γ's are given

Since it can be derived that using the bilinear transformation $s = \frac{2}{T}\frac{z-1}{z+1}$ (with $T = 2$)

$$H_x(s) = \frac{s-b}{s+b} \quad \text{translates into} \quad H_x(z) = \frac{B - z^{-1}}{1 - Bz^{-1}}$$

and

$$H_y(s) = \frac{s^2 - as + b}{s^2 + as + b} \quad \text{translates into} \quad H_y(z) = \frac{-A - Bz^{-1} + z^{-2}}{1 - Bz^{-1} - Az^{-2}}$$

we can write for the transfer function of the *lowpass* filter (N is odd) in the discrete time domain:

$$H(z) = \frac{H_1(z) + H_2(z)}{2}$$

in which

$$H_1(z) = \frac{B_0 - z^{-1}}{1 - B_0 z^{-1}} \prod_{i=2}^{(N-1)/2} \frac{-A_i - B_i z^{-1} + z^{-2}}{1 - B_i z^{-1} - A_i z^{-2}} \quad \text{for even } i\text{'s only}$$

$$H_2(z) = \prod_{i=1}^{(N-1)/2} \frac{-A_i - B_i z^{-1} + z^{-2}}{1 - B_i z^{-1} - A_i z^{-2}} \quad \text{for odd } i\text{'s only}$$

with

$$B_0 = \gamma_0$$
$$A_i = \gamma_{2i-1}$$
$$B_i = \gamma_{2i}(1 - \gamma_{2i-1})$$

For a *bandpass* filter (N is even), we obtain:

$$H(z) = \frac{H_1(z) - H_2(z)}{2}$$

$$H_1(z) = \prod_{i=1}^{N/2} \frac{-A_i - B_i z^{-1} + z^{-2}}{1 - B_i z^{-1} - A_i z^{-2}} \quad \text{for odd } i\text{'s only}$$

$$H_2(z) = \prod_{i=2}^{N/2} \frac{-A_i - B_i z^{-1} + z^{-2}}{1 - B_i z^{-1} - A_i z^{-2}} \quad \text{for even } i\text{'s only}$$

with

$$A_i = \gamma_{2i-1}$$
$$B_i = \gamma_{2i}(1 - \gamma_{2i-1})$$

Appendix

For two port adapters, the following equations hold

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -\gamma & 1+\gamma \\ 1-\gamma & \gamma \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

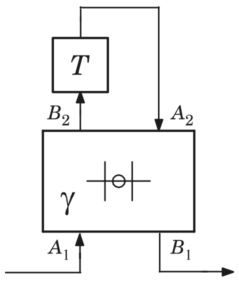

Figure 3: First order Wave Digital allpass section.

For the circuit in Figure 3, with a delay element between output B_2 and input A_2, it can be derived that

$B_1 = -\gamma A_1 + (1+\gamma)A_2,$

$B_2 = (1-\gamma)A_1 + \gamma A_2,$ and

$A_2 = z^{-1}B_2$

from which follows that

$$\frac{B_1}{A_1} = \frac{-\gamma + z^{-1}}{1 - \gamma z^{-1}}$$

that is, the transfer function of a first order allpass section in the discrete time domain.

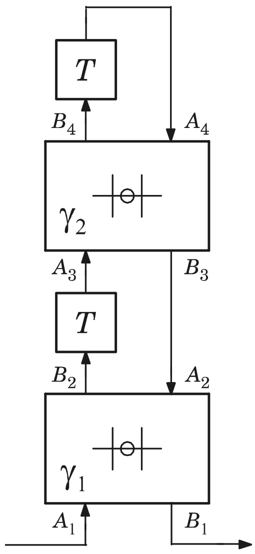

Figure 4: Second order Wave Digital allpass section.

For a setup as in Figure 4, we can find that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -\gamma_1 & 1+\gamma_1 \\ 1-\gamma_1 & \gamma_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\begin{bmatrix} B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} -\gamma_2 & 1+\gamma_2 \\ 1-\gamma_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix}$$

with $A_2 = B_3$, while $A_3 = z^{-1}B_2$ and $A_4 = z^{-1}B_4$

This leads to

$$\frac{B_1}{A_1} = \frac{-\gamma_1 - \gamma_2(1-\gamma_1)z^{-1} + z^{-2}}{1 - \gamma_2(1-\gamma_1)z^{-1} - \gamma_1 z^{-2}}$$

what can be recognized as the transfer function of a second order allpass section in the discrete time domain.