Calculating k for a Cauer filter when the filter order (N), the maximum ripple in the passband $(A_{\max})$ and the minimum ripple in the stopband $(A_{\min})$ are known.

From $A_{\max}$ and $A_{\min}$ we can calculate k_1 and $k_1^{'}$ according to

$$k_1 = \sqrt{\frac{10^{\,0.1A_{\max}} - 1}{10^{\,0.1A_{\min}} - 1}} \quad \text{and} \quad k_1^{'} = \sqrt{1 - k_1^2},$$

which gives us also K_1 and $K_1^{'}$ (preferably by using the AGM-method).
The goal now, is to find k such that the K and $K^{'}$, resulting from that k, satisfy the equality $\dfrac{K^{'}}{K} = \dfrac{K_1^{'}}{NK_1}$.

Therefore, we use the approximation for sn using theta-functions [1]:

$$sn(z,k) = \frac{1}{\sqrt{k}} \frac{\theta_1(\frac{z}{2K}, q)}{\theta_0(\frac{z}{2K}, q)} \tag{1}$$

in which $q = e^{-\pi\frac{K^{'}}{K}}$, or $q = e^{-\pi\frac{K_1^{'}}{NK_1}}$, and

$$\theta_1(\frac{z}{2K}, q) = 2\sqrt[4]{q}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)} \sin\left[(2m+1)\,\frac{\pi z}{2K}\right]$$

$$\theta_0(\frac{z}{2K}, q) = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2} \cos\left(2m\frac{\pi z}{2K}\right).$$

Knowing that $\mathrm{sn}(K,k) = 1$ for all k, we will rewrite the θ-equations for $z = K$:

$$\theta_1(\frac{1}{2}, q) = 2\sqrt[4]{q}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)} \sin\left[(2m+1)\,\frac{\pi}{2}\right] \;=\; 2\sqrt[4]{q}\sum_{m=0}^{\infty} q^{m(m+1)}$$

$$\theta_0(\frac{1}{2}, q) = 1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2} \cos(m\,\pi) \;=\; 1 + 2\sum_{m=1}^{\infty} q^{m^2},$$

while both $\sin\left[(2m+1)\,\frac{\pi}{2}\right]$, as well as $\cos(m\,\pi)$ reduce to another $(-1)^m$.

Thus

$$1 = \frac{1}{\sqrt{k}} \frac{2\sqrt[4]{q} \sum_{m=0}^{\infty} q^{m(m+1)}}{1 + 2\sum_{m=1}^{\infty} q^{m^2}} \tag{2}$$

or

$$k = 4\sqrt{q} \left(\frac{\sum_{m=0}^{\infty} q^{m(m+1)}}{1 + 2\sum_{m=1}^{\infty} q^{m^2}} \right)^2 \tag{3}$$

When we choose $m_{\max} = 3$ instead of ∞, we can calculate k with

$$\boxed{k = 4\sqrt{q} \left(\frac{1 + q^2 + q^6 + q^{12}}{1 + 2q + 2q^4 + 2q^9} \right)^2 \quad \text{in which} \quad q = e^{-\pi \frac{K_1^{'}}{NK_1}}}$$

Some reflections on the accuracy of the approximation of k

To evaluate the usefulness of the approximation of k, we will calculate $\frac{K^{'}}{K}$ as a function of k_{ref} using the AGM-method, and from these $\frac{K^{'}}{K}$ recompute the approximated k_{app}. We will perform the approximation using different orders of the θ-functions in the numerator and denominator of (3). Therefore, we will rewrite (3) as

$$k_{app} = 4\sqrt{q} \left(\frac{\sum_{mn=0}^{MN} q^{mn(mn+1)}}{1 + 2\sum_{md=1}^{MD} q^{md^2}} \right)^2 \tag{4}$$

In Figures 1 and 2, the resulting error $|k_{app} - k_{ref}|$ is shown with MN and MD as parameters. Note that the error plots for $MN = MD = 3$ and those for $MN = 4, MD = 3$ completely overlap, so increasing the order of only the numerator seems to be ineffective.

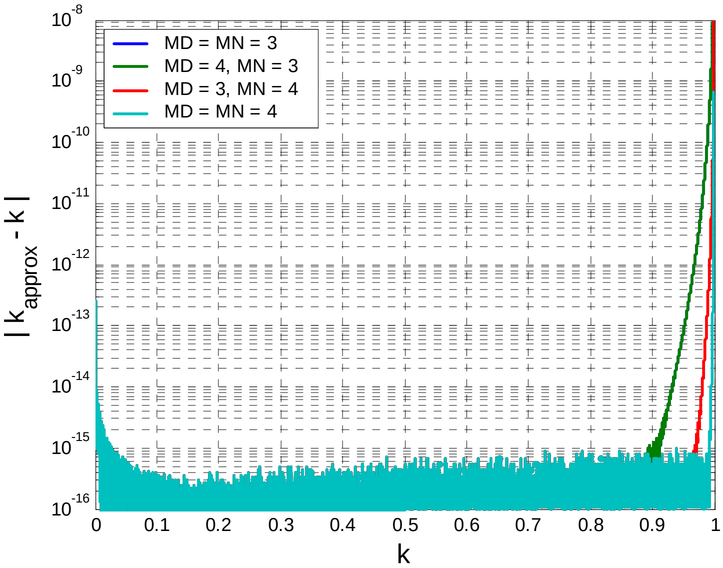

Figure 1: Error for k between 0 and 1

From tables, like those given in Christian & Eisenmann [2], it follows that it will be reasonable to focus on k values between about 0.6 and 0.9 for representing realistic Cauer filters. For those k's, the Figures show that an approximation based on $MN = MD = 3$ will be more than sufficient. When larger errors (in the order of not exceeding 10^{-6}) are allowed, even $MN = MD = 2$ can be tolerated.

For better approximations for $k > 0.9$, it is of course possible to increase MD or even MD and MN depending on the value of $\frac{K^{'}}{K}$ (or $\frac{K_1^{'}}{NK_1}$) resulting from the design parameters. In Figure 3 , $\frac{K^{'}}{K}$ is shown for $0.9 \leq k \leq 1.0$.

H.J. Lincklaen Arriëns, November 2002.

References

[1] A.Antoniou, *Digital Filters: Analysis and Design*, McFraw-Hill Book Company (1979)

[2] E. Christian & E. Eisenmann, *Filter Design Tables and Graphs*, John Wiley & Sons, Inc. (1966)

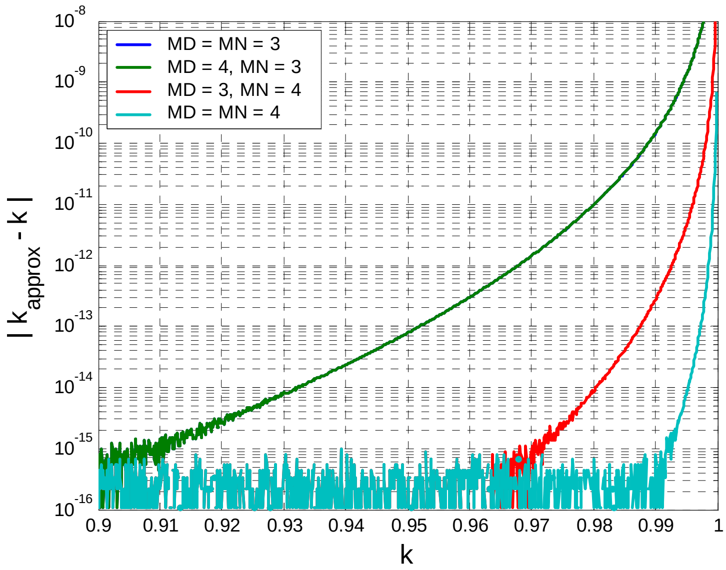

Figure 2: Error for k zoomed in to 0.9 to 1

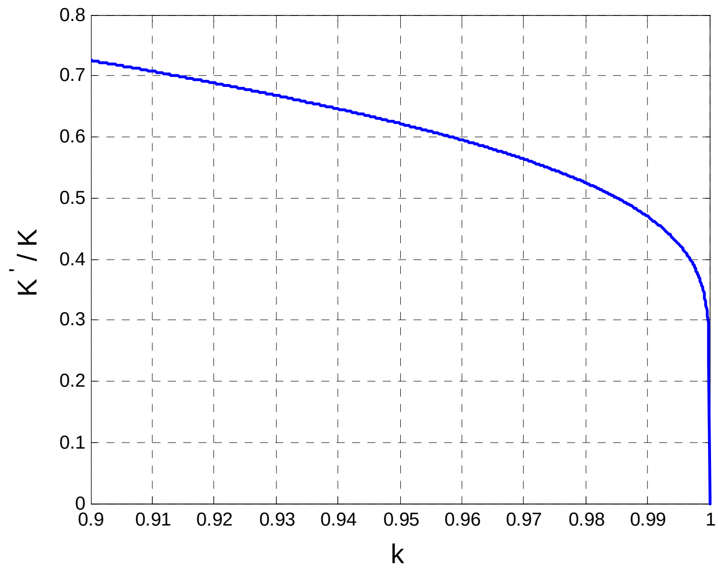

Figure 3: $\frac{K'}{K}$ as a function of k