On the Use of the Sharpe - Chebyshev Rational Function Approximation

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Abstract— **The approximation problem in filter design is well known. For this purpose we have the Butterworth, Chebyshev and Cauer methods, which are all analytical methods. Only the Cauer approximation delivers poles in the stopband and is optimal in terms of selectivity. The poles are directly fixed to the zeros of reflection in the passband and are derived by the use of elliptic functions.**

The Sharpe-Chebyshev approximation method obtains a Chebyshev approximation in the passband together with a free choice of poles in the rest of the complex plane. The Sharpe method appears to be the only analytical method that delivers this and appears not to be very well known.

The free choice of poles in the rest of the complex plane will be shown to be of importance in the area of microwave unit element filter design and in the case of wave digital filter design.

Keywords— **Chebyshev; Sharpe; Wave Digital Filter; Unit-element Filter**

I. INTRODUCTION

The approximation problem in continuous filter design is well known. For this purpose often the Butterworth, Chebyshev and Cauer methods are used, which are all analytical methods. Only the Cauer approximation delivers poles in the stopband and is optimal in terms of selectivity. The poles are directly related to the reflection zeros in the passband and are derived by using elliptic functions.

Lossless ladders are used as prototypes for designing microwave unit-element filters. Because of realizability constraints it is needed to add unit-elements as all-pass sections at both ends of the ladder, shift these unit-elements into the ladder network by using Kuroda identities to make sure the ladder becomes realizable. However, adding unit-elements causes the lossless ladder to become of higher order without having the unit-elements contributing to the filter amplitude characteristic. This shifting into the ladder becomes problematic when resonant circuits are being encountered.

We show by an example that the Sharpe rational function approach can remedy this. The Sharpe method delivers optimal Chebyshev approximation in the passband, directly related to the total order of the ladder network including the used unit-elements.

Lossless ladders are also used for prototyping wave digital filters [1]. These filters have certain parallelism properties inherent in the arithmetical calculations involved. By adding unit-elements to the lossless ladder we can change these parallelism properties. So far, this suffered from the same drawback, the unit-elements increased the order without contributing to the filter amplitude characteristics. Here also, the Sharpe rational function approach can give optimal Chebyshev approximation in the passband taking the unit-elements into account.

II. DERIVATION OF THE CHEBYSHEV RATIONAL FUNCTION

The Sharpe method [2] starts by observing that the well-known Chebyshev polynomial of order 2 has a plot similar to figure 1 and this frequency behavior can also be written as in (1), assuming for the time being that a_i is a real constant that can be chosen freely:

$$\cos\delta_i = \frac{(2a_i^2+1)\omega^2 - a_i^2}{\omega^2 + a_i^2} \tag{1}$$

Equation (1) in figure 1 behaves similar to the Chebyshev polynomial $T_2(\omega^2)$, with

$$T_{2m}(\omega^2) = \cos 2m\Phi, \quad \Phi = \cos^{-1}\omega \tag{2}$$

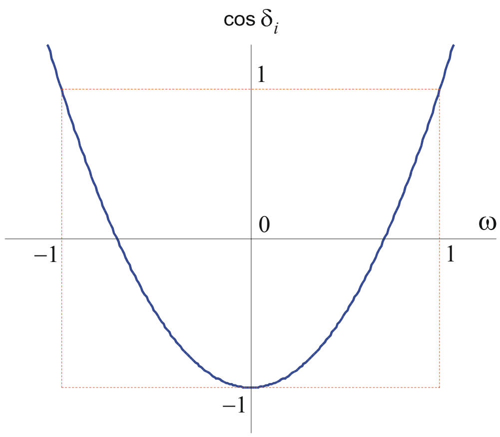


Figure 1 Example plot of $\cos\delta_i$, showing the strong resemblance with $T_2(\omega^2)=\cos 2\Phi$.

Corresponding to (1) we can find that:

$$\sin\delta_i = \frac{2a_i\omega\sqrt{\left(a_i^2+1\right)\left(1-\omega^2\right)}}{\omega^2+a_i^2} \tag{3}$$

The angle δ_i varies in the same manner as 2Φ in the interval $-1 \le \omega \le 1$ if the convention is made that the sign of $\sqrt{1-\omega^2}$ is positive and the sign of $\sqrt{a^2+1}$ is the same as the sign of a_i. Both δ_i and 2Φ are symmetric about π and have the same end points. The above convention with regard to signs insures that the derivative $(d\delta_i)/(d\omega)$ is always negative which makes δ_i a monotonically decreasing function of ω in the interval $-1 \le \omega \le 1$.

We now form the function

$$F\left(\omega^2\right)=\cos\left(2m\Phi+\delta_1+\delta_2+\ldots+\delta_n\right) \tag{4}$$

.

The variation of $F(\omega^2)$ with ω is illustrated in figure 2 for the case of $m+n=3$.

$F(\omega^2)$ can be expanded in the form

$$F\left(\omega^2\right)=\frac{C_0\omega^{2(m+n)}+C_2\omega^{2(m+n)-2}+\cdots+C_{2(m+n)}}{\prod_{i=1}^{n}\left(\omega^2+a_i^2\right)} \tag{5}$$

by noting that

$$F\left(\omega^2\right)=\frac{1}{2}\left[A\prod_{i=1}^{n}B_i + A^*\prod_{i=1}^{n}B_i^*\right] \tag{6}$$

where

$$A=\left(\cos\Phi+j\sin\Phi\right)^{2m},\quad A^*=\left(\cos\Phi-j\sin\Phi\right)^{2m}$$
$$B_i=\left(\cos\delta_i+j\sin\delta_i\right),\quad B_i^*=\left(\cos\delta_i-j\sin\delta_i\right)$$

and

$$F\left(\omega^2\right)=\frac{(X)^{2m}\prod_{i=1}^{n}Z_i}{2\prod_{i=1}^{n}\left(\omega^2+a_i^2\right)}+\frac{\left(X^*\right)^{2m}\prod_{i=1}^{n}Z_i^*}{2\prod_{i=1}^{n}\left(\omega^2+a_i^2\right)} \tag{7}$$

with

$$X,\,X^*=\omega\pm j\sqrt{1-\omega^2}$$
$$Z_i,\,Z_i^*=\left\{\left(2a_i^2+1\right)\omega^2-a_i^2\right\}\pm j2a_i\omega\sqrt{\left(a_i^2+1\right)\left(1-\omega^2\right)}$$

The function $F(\omega^2)$ is already a function of ω^2 which means that there is no need for it to be squared to be used to generate the power scattering transfer function of a lossless filter terminated in resistances as follows:

$$|S_{21}|^2=\frac{1}{1+\varepsilon F'(\omega^2)}\quad \text{with}\quad F'=\frac{F+1}{2} \tag{8}$$

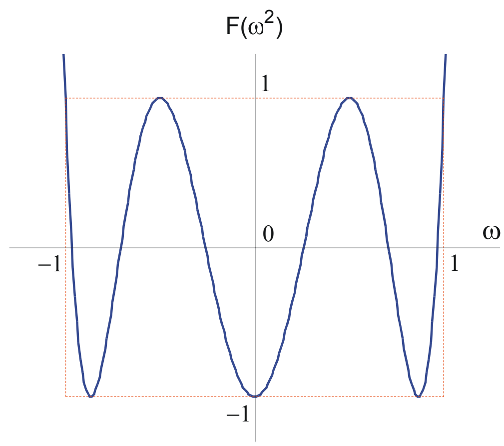


Figure 2 Example of $F(\omega^2)$ for the case of $m+n=3$

III. AN EXAMPLE

As a simple example to demonstrate the results that can be obtained with the Sharpe method we have chosen a 3rd order Cauer normalized lowpass from a table [3]:

$C03,\ \rho = 25\,\%,\ A_{\max} = 0.28029$ dB,
$A_{\min} = 20.58$ dB, $\varepsilon = 0.6667$
Attenuation pole $\omega_p = 1.74228639$, attenuation zero $\omega_\alpha = 0.892921$, start of the stopband at $\omega = 1.55573$.

The corresponding lossless ladder network can be found from other table books.[4] and is given in figure 3 as filter A. We will show here how it can be generated with the Sharpe method. We choose $m = 1$ and $n = 2$ in (4) because we have two poles: $+j\omega_p$ and $-j\omega_p$.

For the purpose of clarifying the generation of the functions F_3, F_4 and F_5 from (7), corresponding to the filters A, B and C of figure 3 we define:

$$X,\ X^* = \omega \pm j\sqrt{1-\omega^2}$$

$$Z_i,\ Z_i^* = \left\{\left(2a_i^2+1\right)\omega^2 - a_i^2\right\} \pm j2a_i\omega\sqrt{\left(a_i^2+1\right)\left(1-\omega^2\right)}$$

with $a_i = \pm j\omega_p$.

For including a unit-element, which is in fact a special case of Z with $a_i = +1$, we define

$$Y_i,\ Y_i^* = \left(3\omega^2-1\right) \pm j2\omega\sqrt{2\left(1-\omega^2\right)}$$

We obtain for the generation of filter A, from (7):

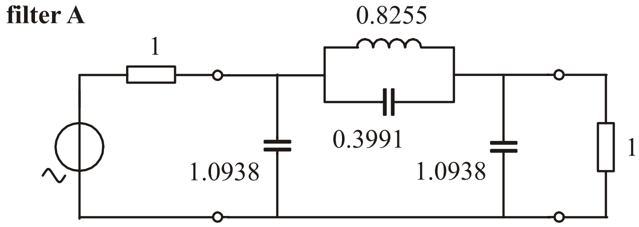
$$F_3\left(\omega^2\right) = \frac{(X)^2(Z)^2 + \left(X^*\right)^2\left(Z^*\right)^2}{2\left(\omega^2-\omega_p^2\right)^2} \tag{9}$$

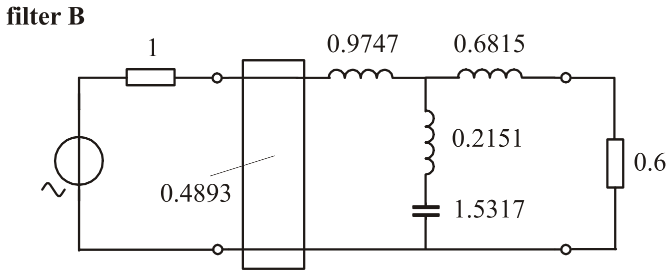
$$F_3' = \frac{F_3+1}{2} = \frac{C_3\,\omega^2\left(\omega^2-\omega_\alpha^2\right)^2}{\left(\omega^2-\omega_p^2\right)^2} \tag{10}$$

with $\omega_\alpha = 0.892921$, $\omega_p = 1.74228639$, and

$C_3 = 100.855$, which is chosen such that $F_3'{}_{(\omega=1)} = 1$.

We are now able to calculate S_{21} as in (8). From this we can calculate the impedance Z_{in} as seen from the source. Ladder synthesis of Z_{in} generates the lossless ladder filter A, which indeed exactly matches the element values given in [4].





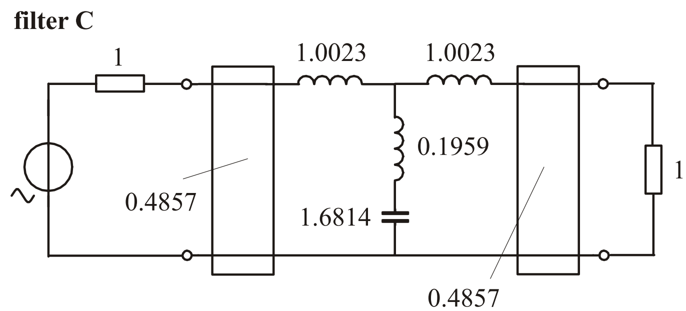


Figure 3 Ladder structures and element values of the filter without unit-elements (A), with one unit-element (B) and with two unit-elements (C).

We now add a unit-element to the filter, keeping $A_{\max}$ and ω_p the same value. Instead of adding it as an all-pass, we now add a special pole to F_3 (now $m = 1$ and $n = 3$ in (4)) and obtain as regular 4th order approximations F_4 and F_4':

$$F_4\left(\omega^2\right) = \frac{(X)^2(Y)(Z)^2 + \left(X^*\right)^2\left(Y^*\right)\left(Z^*\right)^2}{2\left(\omega^2+1\right)\left(\omega^2-\omega_p^2\right)^2} \tag{11}$$

$$F_4' = \frac{F_4+1}{2} = \frac{C_2\left(\omega^2-\omega_{\alpha 1}^2\right)\left(\omega^2-\omega_{\alpha 2}^2\right)^2}{\left(\omega^2+1\right)\left(\omega^2-\omega_p^2\right)^2} \tag{12}$$

in which now $\omega_{\alpha 1} = 0.3813197$, $\omega_{\alpha 2} = 0.927936$ and $C_2 = 587.8276$.

Proceeding in the same manner as before, we obtain filter B.

Finally in this example, we add a second unit-element to filter A, which means a 5th order circuit. Again, we leave $A_{\max}$ and ω_p the same value. We add again a special pole to F_4 (now $m=1$ and $n=4$ in (4)) and obtain as regular 5th order approximations F_5 and F_5':

$$F_5\left(\omega^2\right) = \frac{(X)^2(Y)^2(Z)^2 + \left(X^*\right)^2\left(Y^*\right)^2\left(Z^*\right)^2}{2\left(\omega^2+1\right)^2\left(\omega^2-\omega_p{}^2\right)^2} \tag{13}$$

$$F_5' = \frac{F_5+1}{2} = \frac{C_5\,\omega^2\left(\omega^2-\omega_{\alpha 3}{}^2\right)\left(\omega^2-\omega_{\alpha 4}{}^2\right)^2}{\left(\omega^2+1\right)^2\left(\omega^2-\omega_p{}^2\right)^2} \tag{14}$$

in which now $\omega_{\alpha 3} = 0.561735$, $\omega_{\alpha 4} = 0.94783$ and $C_5 = 3426.11$.

Again proceeding in the same manner as before, we obtain filter C, which is again symmetrical as expected.

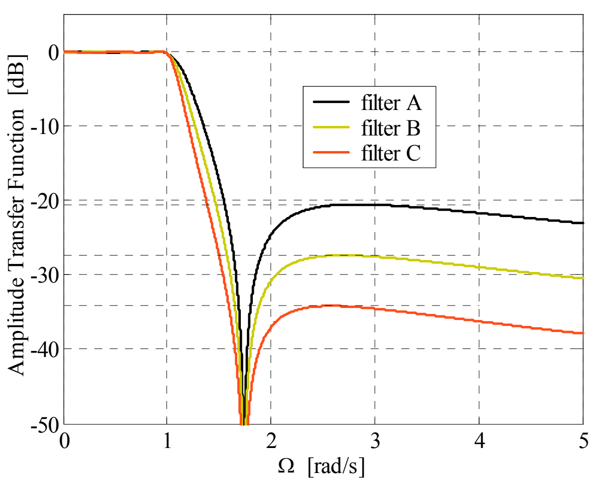


Figure 4 Amplitude transfer functions of the filter without unit-elements (A), with one unit-element (B) and with two unit-elements (C).

It needs to be stressed here, that the sequence in which the unit-elements in filter B and C have been extracted, is not the only possible one.

The resulting attenuation functions for each of the 3 circuits is shown in figure 4 indicating that the unit-elements are indeed contributing considerably to the characteristic. Figure 5 gives detail about the passband and figure 6 gives more detail about the stopband.

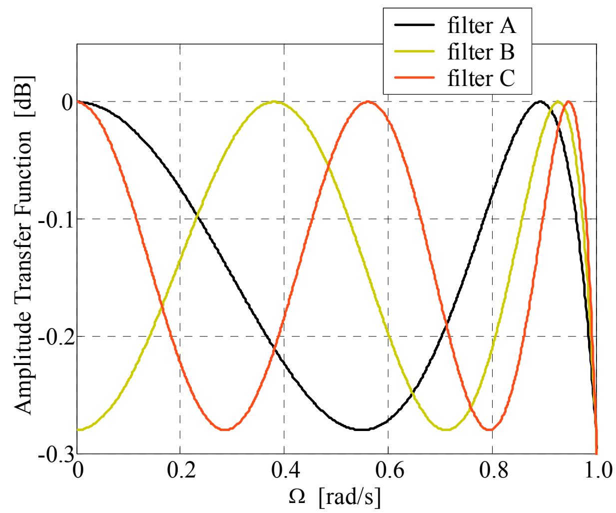


Figure 5 Passband of the same filters as used in Figure 4.

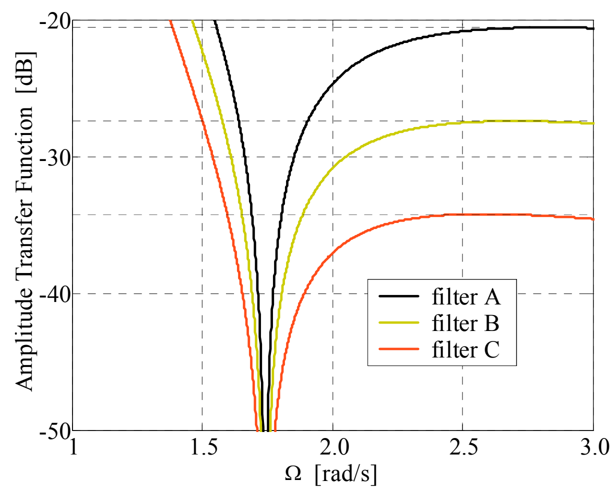


Figure 6 Part of the stopband of the same filters as used in Figures 4 and 5.

Instead of having the unit-elements contributing to the stopband attenuation by keeping ω_p constant, we might also decide to adjust ω_p such that the stopband $A_{\min}$ remains the same, leaving the unit-elements to contribute to the increase in selectivity as is shown in figure 7.

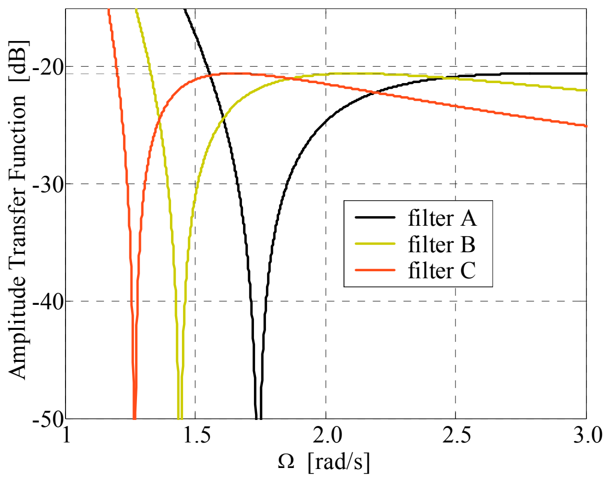


Figure 7 Illustrating the increase in selectivity by adding unit-elements when the stopband attenuation is kept constant.

IV. CONCLUSIONS AND DISCUSSION

We have shown by a simple example that the Sharpe method enables us to force unit-elements to contribute to the filter characteristic instead of adding only delay. Shifting unit-elements into Cauer type ladders is no longer a problem because they can be extracted from Z_{in} at any desired moment during the synthesis process.

The method delivers optimal Chebyshev behavior in the passband with a free choice of poles in the stopband that can be used to design optimal and realizable microwave unit-element filters.

The method also generates wave-digital filter prototypes with different parallelism properties.

It appears that the method is not well known as it is mentioned very little in the filter literature. The method, however, is very powerful. It is being described in Vlach [5] where the method is transformed to another complex variable with the consequence that:

1. more accuracy is obtained because the calculations become simpler,
2. bandpass filters can be generated (also containing unit-elements if needed) with a free choice of attenuation poles.

Rhodes [6] only mentions the method in a short appendix in his book.

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